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Almost Gibbsian versus weakly Gibbsian measures

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Abstract

We consider two possible extensions of the standard definition of Gibbs measures for lattice spin systems. When a random field has conditional distributions which are almost surely continuous (almost Gibbsian field), then there is a potential for that field which is almost surely summable (weakly Gibbsian field). This generalizes the standard Kozlov theorems. The converse is not true in general as is illustrated by counterexamples.

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1. Situation

The standard definition of a Gibbs random field for lattice spin systems starts from an interaction potential. With this potential one associates a local Hamiltonian H_Λ^ω (sum of potentials) in a finite volume Λ with boundary condition ω outside Λ. The local Hamiltonian determines the finite volume Gibbs measure μ_Λ^ω in Λ with boundary condition ω outside Λ via the classical Boltzmann–Gibbs formula. The infinite volume Gibbs measures are then defined as those measures on configuration space for which the conditional probabilities in Λ with ω fixed outside Λ are precisely the finite volume measures μ_Λ^ω. In the standard formalism, the sum of potentials that make up the Hamiltonian is supposed to converge uniformly in the configuration. This implies that the local Hamiltonian is continuous as a function of the boundary condition. It implies further that the conditional probabilities of the (infinite volume) Gibbs measure have a continuous version. The theorems of Kozlov (1974) and Sullivan (1973) deal with the converse: given a measure that allows a continuous (and strictly positive) version of its conditional probabilities, there exists a potential, associated to this measure in the way just described, which is uniformly convergent.

In the last 10 years, there has been an intensive study of the limits of the Gibbsian formalism. Physically, relevant non-Gibbsian fields were constructed. They

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have mostly appeared as image measures under renormalization group transformations of Gibbs measures. In other cases (like for stationary measures of interacting particle systems modeling some non-equilibrium situation), it is not at all clear whether Gibbs measures appear and in some cases this was disproven. The situation as in 1993 was summarized in van Enter et al. (1993) and we refer to this article for further background. Even before, however, Dobrushin stressed that one should allow for more general definitions of a Gibbs state than is usually done. He made the analogy with unbounded spins where the summability of the potential cannot be understood uniformly. One should look for the 'good' configurations on which the usual Gibbsian game can be played. This was illustrated in his last conference talk where he showed how to give a Gibbsian characterization of a non-Gibbsian state (Dobrushin, 1995; Dobrushin et al., 1997). This program was carried further in Bricmont et al. (1998) and Maes et al. (1997). Many questions remain, however (see e.g. Dobrushin et al., 1997; Lörinczi et al., 1997a, b). In short, what are possible generalizations of the standard Gibbsian framework, how do they relate, how typical are they and what remains of the Gibbs formalism?

In the present paper, we take a closer look at two of the main generalizations that have appeared. We call them almost versus weakly Gibbsian measures. Almost Gibbsian measures have a version of their conditional distributions which is almost surely continuous. Weakly Gibbsian states allow for an almost surely absolutely convergent potential. Almost Gibbsianness looks for a large set of 'good' configurations defined in terms of continuity points of the conditional distributions, while weak Gibbsianness asks for a large set of 'good' configurations on which the potential satisfies a certain summability. We are therefore reminded of the Kozlov–Sullivan theorems and observe that almost Gibbs implies weakly Gibbs. The converse is not true, as will be illustrated by a counterexample. More generally, the paper tends to add some more structure on the 'road' from Gibbsian to non-Gibbsian measures.

2. Potentials and specifications

In this section we introduce some basic notions related to lattice spin systems, used throughout the paper.

We consider the regular d-dimensional lattice $\mathbb{Z}^d$ and denote by $\mathscr{L} := \{\Lambda, |\Lambda| < \infty\}$ the set of finite subsets of $\mathbb{Z}^d$. The complement of a set $\Lambda \subset \mathbb{Z}^d$ is $\Lambda^c = \mathbb{Z}^d \setminus \Lambda$. Infinite volume limits of a function $a : \mathscr{L} \to \mathbb{R}$

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} a(\Lambda) = a \tag{2.1}$$

must be understood as saying that for all $\varepsilon > 0$ there is a $\Lambda(\varepsilon) \in \mathscr{L}$ so that $|a(\Lambda) - a| < \varepsilon$ whenever $\Lambda(\varepsilon) \subset \Lambda \in \mathscr{L}$. This is applied for infinite sums which should be read as

$$\sum_M u(M) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \sum_{M \subset \Lambda} u(M) \tag{2.2}$$

and Eq. (2.1) must be applied to $a(\Lambda) = \sum_{M \subset \Lambda} u(M)$. Such sums can be convergent (i.e. the limit in Eq. (2.2) is a finite number) without being *absolutely* summable and there is no explicit fixing of a sequence of increasing volumes along which the limits ought to be taken.

The state space is $\Omega := W^{\mathbb{Z}^d}$ where $W := \{1,\dots,q\}$ is a finite set and its elements (= configurations) are denoted by greek letters $\eta, \omega, \sigma, \xi, \dots$. The value of ω at a site $i \in \mathbb{Z}^d$ is written as $\omega(i)(\in W)$. On Ω we have the natural action of translations τ_a, $a \in \mathbb{Z}^d$ defined by $\tau_a \eta(i) := \eta(i-a)$, $i \in \mathbb{Z}^d$. The restriction of ω to a subset $M \subset \mathbb{Z}^d$ is ω_M and for $\omega, \eta \in \Omega$ we define $\omega_M^\eta \in \Omega$ by

$$\begin{aligned} \omega_M^\eta(k) &:= \omega(k), \quad k \in M, \\ \omega_M^\eta(k) &:= \eta(k), \quad k \in M^c. \end{aligned} \tag{2.3}$$

For example, $\omega_M^1(k) = 1$ if $k \in M^c$ and $= \omega(k)$ for $k \in M$. The σ-algebra generated by the evaluation maps X_i, $X_i(\omega) := \omega(i)$, $i \in M$ is written as $\mathscr{F}_M = \sigma\{X_i,\ i \in M\}$. When $M = \mathbb{Z}^d$, we set $\mathscr{F} := \mathscr{F}_{\mathbb{Z}^d}$. The *tail field* σ-algebra $\mathscr{T}^\infty$ is defined as

$$\mathscr{T}^\infty := \bigcap_{\Lambda \in \mathscr{L}} \mathscr{F}_{\Lambda^c}. \tag{2.4}$$

The configuration space Ω is a compact metric space in the product topology. We call a function f on Ω local if it depends only on a finite number of coordinates, i.e. there is a $\Lambda \in \mathscr{L}$ such that $f(\eta) = f(\zeta)$ whenever $\eta_\Lambda = \zeta_\Lambda$.

Definition 2.1. (1) $f : \Omega \to \mathbb{R}$ is continuous at a point $\omega \in \Omega$ if

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \sup_{\theta \in \Omega} |f(\omega) - f(\omega_\Lambda^\theta)| = 0. \tag{2.5}$$

(2) Let $\theta \in \Omega, f : \Omega \to \mathbb{R}$ is continuous in the direction θ at a point $\omega \in \Omega$ if

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} |f(\omega) - f(\omega_\Lambda^\theta)| = 0. \tag{2.6}$$

(3) $f : \Omega \to \mathbb{R}$ is continuous (in the direction θ) if it is continuous at every point $\omega \in \Omega$ (in the direction θ).

Every continuous function is a uniform limit of local functions (by the Stone–Weierstrass theorem). A continuous function is continuous in every direction, but a function can be continuous in every direction (and bounded), yet can fail to be continuous.

2.1. Specifications

Definition 2.2. A specification Γ on $\mathscr{L}$ is a family of probability kernels $\Gamma = \{\gamma_\Lambda, \Lambda \in \mathscr{L}\}$ on $(\Omega, \mathscr{F})$, such that the following hold:
(1) $\gamma_\Lambda(\cdot|\omega)$ is a probability measure on $(\Omega, \mathscr{F})$ for all $\omega \in \Omega$;
(2) $\gamma_\Lambda(F|\cdot)$ is $\mathscr{F}_{\Lambda^c}$-measurable for all $F \in \mathscr{F}$;
(3) $\gamma_\Lambda(F|\omega) = 1_F(\omega)$ if $F \in \mathscr{F}_{\Lambda^c}$;
(4) $\gamma_{\Lambda_2}\gamma_{\Lambda_1} = \gamma_{\Lambda_2}$ if $\Lambda_1 \subset \Lambda_2$.

The last property (consistency) is most important in characterizing equilibrium. One should also remember that $\gamma_\Lambda(\sigma_\Lambda^\omega|\omega)$, which has to be thought of as the probability to find σ in Λ given ω outside, is a function defined (pointwise) for every $\sigma,\omega\in\Omega$ (which only depends on σ_Λ and ω_{Λ^c}).

In this paper we restrict ourselves to specifications that are *uniformly non-null*, i.e. $\forall\Lambda\in\mathscr{L}$ $\exists$ a constant $m_\Lambda>0$ such that $\forall\omega\in\Omega$

$$\gamma_\Lambda(\omega|\omega)\geqslant m_\Lambda. \tag{2.7}$$

A specification Γ is said to be *translation invariant* if $\forall a\in\mathbb{Z}^d$, $\forall\Lambda\in\mathscr{L},\forall\omega\in\Omega$ and for all bounded functions f

$$\gamma_\Lambda(f\circ\tau_a|\omega)=\gamma_{\Lambda+a}(f|\tau_a\omega). \tag{2.8}$$

Definition 2.3. A probability measure μ on $(\Omega,\mathscr{F})$ is consistent with a specification Γ (notation: $\mu\in\mathscr{G}(\Gamma)$) if $\forall\Lambda\in\mathscr{L}$ and for every continuous function f on Ω:

$$\mu\circ\gamma_\Lambda(f)=\mu(f), \tag{2.9}$$

where $\mu\circ\gamma_\Lambda(f)=\int\mu(\mathrm{d}\omega)\gamma_\Lambda(f|\omega)$.

We say that a probability measure is uniformly non-null if it is consistent with a uniformly non-null specification.

Remark. We will also say that a specification Γ is consistent with the probability measure μ. Notice that $\mathscr{G}(\Gamma)$ is a convex set which may be empty.

We define (measurable) sets of continuity points for a specification Γ as follows:

$$\Omega_\Gamma:=\{\omega:\forall\Lambda\in\mathscr{L},\ \forall F\in\mathscr{F}_\Lambda,\ \gamma_\Lambda(F|\cdot)\text{ is continuous at }\omega\}, \tag{2.10}$$

$$\Omega_\Gamma^\theta:=\{\omega:\forall\Lambda\in\mathscr{L},\ \forall F\in\mathscr{F}_\Lambda,\ \gamma_\Lambda(F|\cdot)\text{ is continuous at }\omega\text{ in the direction }\theta\}. \tag{2.11}$$

Note that for a uniformly non-null specification, the continuity of $\gamma_\Lambda(F|\cdot)$ for all sets $\Lambda\in\mathscr{L}$ follows from the continuity of the single site probabilities $\gamma_{\{i\}}(\sigma|\cdot)$. The same is true for continuity in the direction θ.

For a specification that is uniformly non-null, the set Ω_Γ of continuity points is in the tail field. Hence if $\Omega_\Gamma\neq\{\emptyset,\Omega\}$ then both Ω_Γ and Ω_Γ^c are topologically dense subsets of Ω (every tailfield set is dense in Ω). Because a function which is continuous on a dense set can only be discontinuous on a set of first Baire category (cf. Boas, 1960, pp. 101–102) we can conclude that Ω_Γ^c is necessarily a set of first Baire category and hence $\Omega_\Gamma=\Omega\backslash\Omega_\Gamma^c$ is a set of second Baire category. This means that we automatically have that Ω_Γ is larger than Ω_Γ^c in a topological sense. This fact does of course not give information about the measure of Ω_Γ and Ω_Γ^c; we can still have $\mu(\Omega_\Gamma)=0$ and $\mu(\Omega_\Gamma^c)=1$ for some $\mu\in\mathscr{G}(\Gamma)$.

2.2. Potentials

Definition 2.4. A potential U is a real valued function on $\mathscr{L} \times \Omega$

$$U : \mathscr{L} \times \Omega \to \mathbb{R} \tag{2.12}$$

such that $U(A, \cdot) \in \mathscr{F}_A$ for all $A \in \mathscr{L}$ (put $U(\emptyset, \cdot) = 0$).

$U(A, \sigma)$ represents the interaction between spins in A for a configuration σ. A potential U is *translation invariant* if $\forall A \in \mathscr{L},\ a \in \mathbb{Z}^d,\ \eta \in \Omega$

$$U(A, \eta) = U(A + a, \tau_a \eta). \tag{2.13}$$

We say that a potential U is a *vacuum potential* with vacuum θ if $U(A, \omega) = 0$ whenever $\omega(x) = \theta(x)$ for some $x \in A$.

To be useful a potential has to satisfy some summability condition, i.e. the interaction of a finite piece of the lattice with the rest of the world must be finite in some sense.

In the traditional theory one requires that

$$\sum_{A \cap \Lambda \neq \emptyset} \sup_{\omega} |U(A, \omega)| < \infty. \tag{2.14}$$

For our purposes we need more general (pointwise) notions of convergence.

Definition 2.5. A potential U is convergent at ω if for all $\Lambda \in \mathscr{L}$ the local Hamiltonian

$$H_\Lambda^U(\omega) := \sum_{A \cap \Lambda \neq \emptyset} U(A, \omega) \tag{2.15}$$

is well-defined (i.e. the sum is convergent in the sense of Eq. (2.2)).

We still need the following definitions.

Definition 2.6. A potential U is absolutely convergent at $\omega \in \Omega$ if for all $\Lambda \in \mathscr{L}$

$$\sum_{A \cap \Lambda \neq \emptyset} |U(A, \omega)| < \infty. \tag{2.16}$$

If we have for ω that the local Hamiltonian $H_\Lambda^U(\eta_\Lambda^\omega)$ is well defined for all $\eta \in \Omega$, then we can define the finite volume Gibbs measures

$$\mu_\Lambda^U(\eta|\omega) := \frac{1_{[\eta = \omega \text{ on } \Lambda^c]}}{Z_\Lambda(\omega_{\Lambda^c})} \exp[-H_\Lambda^U(\eta_\Lambda^\omega)], \quad \eta \in \Omega \tag{2.17}$$

with fixed boundary condition ω. Here $Z_\Lambda(\omega_{\Lambda^c})$ is the normalization. We did not include here the usual pre-factor β (inverse temperature) nor did we add any specific a priori measure (other than the counting measure) on the single-site state space. Both (temperature and a priori weights) are supposed to be contained in the potential.

Definition 2.7. A potential U is compatible at $\omega \in \Omega$ with a specification Γ if for all $\Lambda \in \mathscr{L}$, and for all $\eta \in \Omega$

$$\mu_\Lambda^U(\eta|\omega) = \gamma_\Lambda(\eta|\omega). \tag{2.18}$$

Remark. If there exists a *vacuum potential* V_Γ^θ with vacuum θ, compatible with a given specification Γ then this V_Γ^θ is unique and is given by

$$V_\Gamma^\theta(A,\omega) = -\sum_{\Lambda \subset A} (-1)^{|A\setminus\Lambda|} \log \frac{\gamma_\Lambda(\omega_\Lambda^\theta|\theta)}{\gamma_\Lambda(\theta|\theta)}. \tag{2.19}$$

Note, however, that

(1) the potential constructed in expression (2.19) above does not need to be convergent in general and

(2) even when it is convergent it is still possible that this V_Γ^θ is *not* compatible with Γ (cf. Proposition 4.1).

The vacuum potential is necessarily translation invariant if Γ is and θ is constant.

3. Pointwise Kozlov's theorems

We give here our generalizations of the results of Kozlov (1974). We omit the proofs since they are straightforward extensions of the proofs in Kozlov (1974), replacing uniform by pointwise formulations. In what follows, we always have a specification Γ as in (Definition 2.2) with corresponding sets of continuity points (2.10) and (2.11).

Theorem 3.1 (Pointwise Kozlov result). *Let Γ be a specification and $\theta \in \Omega$. Then we have*

- *for all $\omega \in \Omega_\Gamma^\theta$, the vacuum potential V^θ with vacuum θ (as in* Eq. (2.19)) *is convergent at ω (as in Definition* 2.5) *and is compatible with Γ at ω,*
- *there exists a potential $U = U^\theta$ which is absolutely convergent (as in Definition* (2.6)) *and compatible with Γ at every $\omega \in \Omega_\Gamma^\theta$.*

Remark. In the case of a translation invariant specification Γ, and θ constant, the potential U of Theorem 3.1 is not necessarily translation invariant, whereas the vacuum potential V_Γ^θ is translation invariant but not necessarily *absolutely* convergent. In order to obtain a translation invariant potential which is absolutely convergent (as in Definition 2.6) we need supplementary conditions. For $x \in \mathbb{Z}^d$ and real r, put $B_x^+(r) := \{y \in \mathbb{Z}^d : y \geqslant x \text{ and } |x-y| \leqslant r\}$ where $|x-y| := \max_{j=1,\dots,d} |x^j - y^j|$ and $y \geqslant x$ if $y^1 > x^1$ or $y^j = x^j$ for $j = 1,\dots,k < d$ and $y^{k+1} > x^{k+1}$. Define

$$\phi_x(r,\omega) := |\gamma_{\{x\}}(\omega|\omega) - \gamma_{\{x\}}(\omega^\theta_{B_x^+(r)}|\omega^\theta_{B_x^+(r)})|, \tag{3.1}$$

where we take θ a constant configuration. We say that ω is a 'good' configuration if $\omega \in K_\Gamma^\theta := \bigcap_x K_x^\theta$ where

$$K_x^\theta = \left\{ \omega \colon \sum_{i=1}^{\infty} \sum_{y \in B_x(2^i)} \phi_y(2^i,\omega) < \infty \right\}. \tag{3.2}$$

Theorem 3.2 (Pointwise Kozlov result). *Let Γ be a translation invariant specification for which the translation invariant set K_Γ^θ of 'good' configurations was defined in* Eq. (3.2). *There exists a translation invariant potential U such that at every $\omega \in K_\Gamma^\theta$, U is absolutely convergent and compatible with Γ.*

In the formulation of the above theorems we did not need to speak about a specific measure; these theorems deal with the relation between specifications and potentials. However, in applications, one wants to understand whether a specific measure allows (what kind of) a potential. In this case, the specification Γ of the theorems above really stands for (a version of) the conditional probabilities of the measure and it acquires a potential via Eqs. (2.17) and (2.18). We give some applications of this in Section 4. The analogue of Theorem 3.2 in the paper of Sullivan (1973) applies for a continuous (and thus uniformly continuous) specification. The author constructs a potential which is translation invariant and absolutely convergent in a weaker sense, but he does not need a condition of the type $K_\Gamma^\theta = \Omega$. His proof relies strongly on the uniform continuity of the specification and is not suited for pointwise generalizations.

4. From almost Gibbsian to non-Gibbsian measures

Definition 4.1. A probability measure μ on $(\Omega, \mathscr{F})$ is Gibbsian if there exists a specification Γ which is uniformly non-null, such that $\mu \in \mathscr{G}(\Gamma)$ and $\Omega_\Gamma = \Omega$.

Definition 4.2. A probability measure μ on $(\Omega, \mathscr{F})$ is

(1) almost Gibbsian if there exists a uniformly non-null specification Γ such that $\mu \in \mathscr{G}(\Gamma)$ and $\mu(\Omega_\Gamma) = 1$,

(2) almost Gibbsian in the direction θ if there exists a uniformly non-null specification Γ such that $\mu \in \mathscr{G}(\Gamma)$ and $\mu(\Omega_\Gamma^\theta) = 1$.

This is equivalent to asking that some version of the conditional probabilities of μ is continuous (continuous in the direction θ) μ-almost surely.

Remark. (1) Of course, every Gibbs measure is almost Gibbsian.

(2) Since for Γ uniformly non-null, Ω_Γ is a set in the tail field, we always have that for extremal $\mu \in \mathscr{G}(\Gamma)$ $\mu(\Omega_\Gamma) = 0$ or $\mu(\Omega_\Gamma) = 1$, i.e. the specification is μ-a.s. continuous or μ-a.s. discontinuous.

Definition 4.3. A probability measure μ on $(\Omega, \mathscr{F})$ is *weakly Gibbsian* if there exists a potential U and a tail field set Ω_U such that

(1) U is absolutely convergent on Ω_U (see Definition 2.6);

(2) $\mu(\Omega_U) = 1$;

(3) $\forall \Lambda \in \mathscr{L}$, $\forall B \in \mathscr{F}_{\Lambda^c}$ and for every bounded measurable function f,

$$\int_B f \, d\mu = \int_B d\mu(\omega) \frac{1}{Z_\Lambda(\omega_{\Lambda^c})} \sum\nolimits_{\sigma_\Lambda} f(\sigma_\Lambda \omega_{\Lambda^c}) e^{-H_\Lambda^U(\sigma_\Lambda \omega_{\Lambda^c})}, \tag{4.1}$$

where

$$Z_\Lambda(\omega_{\Lambda^c}) = \sum_{\sigma_\Lambda} \exp\{-H_\Lambda^U(\sigma_\Lambda \omega_{\Lambda^c})\}. \tag{4.2}$$

Remark. (1) That Ω_U is a tail field set makes $Z_\Lambda(\omega_{\Lambda^c})$ well defined for every $\Lambda \in \mathscr{L}$ and for all $\omega \in \Omega_U$.

(2) In the case of a translation invariant μ one would, of course, like to have U and Ω_U translation invariant. However, even if μ is a Gibbs measure it is unknown whether this is always possible.

(3) Definition 4.3 could be reformulated in terms of specifications as follows: μ is weakly Gibbsian if there exists a potential U, absolutely convergent on a tail field set Ω_U and there exists a specification Γ compatible with U on Ω_U such that $\mu \in \mathscr{G}(\Gamma)$ and $\mu(\Omega_U) = 1$.

(4) It is important to realize that even when $H_\Lambda^U(\omega) = \sum_{A \cap \Lambda \neq \emptyset} U(A, \omega)$ exists on Ω_U, we can not conclude that H_Λ^U is continuous in ω. Therefore weakly Gibbsianness is a much weaker property than almost Gibbsianness. In fact, as we will see in Section 5 there exist weakly Gibbsian measures μ such that for every specification Γ consistent with μ, $\mu(\Omega_\Gamma) < 1$.

(5) If $H_\Lambda^U(\omega) = \sum_{A \cap \Lambda \neq \emptyset} U(A, \omega)$ exists on the whole of Ω (i.e. $\Omega_U = \Omega$), then H_Λ^U is a function of first Baire class (a pointwise limit of continuous functions) and therefore it can only be discontinuous on a set of first Baire category, i.e. a countable union of nowhere dense sets (see Boas, 1960, pp. 99–102). In fact the conclusion that $H_\Lambda^U(.)$ can only be discontinuous on a set of first category is valid for more general Ω_U. The set Ω_U has to be "large" in a topological sense: it has to be such that every open subset (in the restricted topology) is of second category. This means, e.g. that we cannot have that a measure μ is weakly Gibbsian with an everywhere convergent potential and at the same time for every specification Γ consistent with μ, $\Omega_\Gamma = \emptyset$. Of course, it is still possible that $\mu(\Omega_\Gamma) = 0$ for such a measure since there is no relation between topological and measure theoretical "large". In the examples we will illustrate these considerations.

From Section 3, Theorem 3.2, we get the following.

Theorem 4.1. *If μ is almost Gibbsian in the direction θ for some $\theta \in \Omega$ then μ is weakly Gibbsian and we can choose $\Omega_U = \Omega_\Gamma^\theta$ where Γ is such that $\mu \in \mathscr{G}(\Gamma)$.*

For a positive result concerning the converse, we have the following: convergence of the vacuum potential V_Γ^θ does give continuity in the direction θ of the specification Γ, see Eq. (2.19). We fix a specification Γ and a configuration θ:

Proposition 4.1. *The vacuum potential V_Γ^θ is convergent (as in Definition 2.5) at ω and compatible with Γ at ω iff $\omega \in \Omega_\Gamma^\theta$. As a consequence, if μ is weakly Gibbsian for the vacuum potential V_Γ^θ, then μ is almost Gibbsian in the direction θ.*

Proof. From identity (2.19) one obtains

$$\sum_{A \subset L, A \cap \Lambda \neq \emptyset} V_\Gamma^\theta(A, \omega) = -\log \frac{\gamma_\Lambda(\omega_\Lambda | \omega_{L \setminus \Lambda} \theta_{L^c})}{\gamma_\Lambda(\theta_\Lambda | \omega_{L \setminus \Lambda} \theta_{L^c})} \tag{4.3}$$

Proposition 4.1 easily follows from taking the limit $L \uparrow \mathbb{Z}^d$. $\square$

Remark. If μ is almost Gibbsian and translation invariant, we still cannot conclude, in general, that μ is weakly Gibbsian for a translation invariant potential. In order to have this, the extra condition $\mu(K_\Gamma^\theta)=1$ (cf. Eq. (3.2)) must be met by a specification Γ for which $\mu \in \mathscr{G}(\Gamma)$.

Proposition 4.1 can be applied in the context of the paper (Fernandez et al., 1997). The authors consider translation invariant Gibbs measures on $\Omega=\{-1,1\}^{\mathbb{Z}^d}$ which are consistent with a monotone quasilocal specification. By introducing the concept of global specification, they prove that in particular the projections of these Gibbs measures on infinite subsets of $\mathbb{Z}^d$ (which may be non-Gibbsian) are consistent with a monotone right-continuous specification Γ (i.e. $\Omega_\Gamma^\theta=\Omega$ for $\theta\equiv 1$). From Proposition 4.1 it follows that these projected measures are weakly Gibbsian for the vacuum potential V_Γ^1. From Theorem 3.1 it follows that they are weakly Gibbsian for an absolutely convergent (but not necessarily translation invariant) potential. In the special case of the Ising model, Theorem 3.2 can be applied for the projections of the plus (or minus) phase (i.e. K_Γ^θ of Eq. (3.2) is then a set of measure one for the projection), yielding that these projections are weakly Gibbsian for a translation invariant absolutely convergent potential. We will deal with this in a future publication (Maes et al., 1998), see also Dobrushin (1995), Dobrushin et al. (1997) and Maes et al. (1997).

We will now introduce the concept of 'bad' configurations for a probability measure μ. This yields a necessary and sufficient condition on the *finite* conditional distributions of μ for (almost) Gibbsianness of μ. It is in the spirit of Proposition 4.2 in Fernandez et al. (1997), see also van Enter et al. (1996). The motivation to introduce this concept is to be able to detect for a given probability measure μ *essential* discontinuities of the conditional probabilities, i.e. configurations $\sigma\in\Omega$ such that for *every* specification Γ consistent with μ, σ is a point of discontinuity of this specification (i.e. $\sigma\notin\Omega_\Gamma$).

Definition 4.4. A configuration $\xi\in\Omega$ is '*bad*' for μ if there exists a site $i\in\mathbb{Z}^d$ and $\varepsilon>0$ such that for every $\Lambda\in\mathscr{L}$ there are $\Lambda'\supset\Lambda$, $|\Lambda'|<\infty$ and $\omega,\omega'\in\Omega$ such that

$$|\mu(\xi_i|\xi_{\Lambda\setminus i}\omega_{\Lambda'\setminus\Lambda})-\mu(\xi_i|\xi_{\Lambda\setminus i}\omega'_{\Lambda'\setminus\Lambda})|\geqslant\varepsilon. \tag{4.4}$$

The notation in Eq. (4.4) is supposed to be self-explanatory: we ask for the variation in the finite conditional probabilities (of finding the value ξ_i for the spin at site i when the configuration is fixed and equal to ξ in $\Lambda\setminus\{i\}$, while equal to ω or ω' in $\Lambda'\setminus\Lambda$). The proof of the next proposition is trivial:

Proposition 4.2. *Let μ be a probability measure and $\xi\in\Omega$. If ξ is a bad configuration for μ then for all specifications Γ such that $\mu\in\mathscr{G}(\Gamma)$, $\xi\notin\Omega_\Gamma$.*

In the other direction we have:

Proposition 4.3. *A probability measure μ which is uniformly non-null and has no 'bad' configurations is Gibbsian.*

Proof. We show that there exists a specification Γ which is uniformly non-null such that $\mu \in \mathscr{G}(\Gamma)$ and $\Omega_\Gamma = \Omega$. Let $\Lambda_n := [-n, n]^d$. We show that the limit

$$\lim_{n\to\infty} \mu(\xi_0 | \xi_{\Lambda_n}) \tag{4.5}$$

exists for every ξ. Indeed, for $p \geqslant n$

$$\mu(\xi_0|\xi_{\Lambda_n}) = \int \mu(\xi_0|\xi_{\Lambda_n}\omega_{\Lambda_p\setminus\Lambda_n})\mu(\mathrm{d}\omega_{\Lambda_p\setminus\Lambda_n}|\xi_{\Lambda_n}) \tag{4.6}$$

so that

$$\begin{aligned} &|\mu(\xi_0|\xi_{\Lambda_n}) - \mu(\xi_0|\xi_{\Lambda_p})| \\ &\quad = \left| \int \mu(\mathrm{d}\omega_{\Lambda_p\setminus\Lambda_n}|\xi_{\Lambda_n})[\mu(\xi_0|\xi_{\Lambda_n}\omega_{\Lambda_p\setminus\Lambda_n}) - \mu(\xi_0|\xi_{\Lambda_p})] \right|. \end{aligned} \tag{4.7}$$

Since ξ is not a 'bad' configuration

$$\lim_{n,p\to\infty} \sup_{\omega} |\mu(\xi_0|\xi_{\Lambda_n}\omega_{\Lambda_p\setminus\Lambda_n}) - \mu(\xi_0|\xi_{\Lambda_p})| = 0 \tag{4.8}$$

hence $\{\mu(\xi_0|\xi_{\Lambda_n}),\ n \in \mathbb{N}\}$ is a Cauchy sequence. Define

$$\gamma_{\{0\}}(\xi|\xi) := \lim_{n\to\infty} \mu(\xi_0|\xi_{\Lambda_n\setminus 0}) \tag{4.9}$$

and, similarly,

$$\gamma_{\{i\}}(\xi|\xi) := \lim_{n\to\infty} \mu(\xi_i|\xi_{\Lambda_n\setminus i}). \tag{4.10}$$

We prove that $\gamma_{\{0\}}(\xi|\cdot)$ is continuous. Choose $\xi \in \Omega$ and $\varepsilon > 0$. Since ξ is not 'bad', there must be $\Lambda \subset \mathbb{Z}^d$ so that for all $\Lambda' \supset \Lambda$ and every ω, ω'

$$|\mu(\xi_0|\xi_\Lambda\omega_{\Lambda'\setminus\Lambda}) - \mu(\xi_0|\xi_\Lambda\omega'_{\Lambda'\setminus\Lambda})| \leqslant \varepsilon. \tag{4.11}$$

Choose now n such that $\Lambda_n \supset \Lambda$, then for $\tilde{\xi} = \xi$ on Λ

$$|\mu(\xi_0|\xi_{\Lambda_n}) - \mu(\xi_0|\tilde{\xi}_{\Lambda_n})| \tag{4.12}$$

$$= |\mu(\xi_0|\xi_\Lambda\xi_{\Lambda_n\setminus\Lambda}) - \mu(\xi_0|\xi_\Lambda\tilde{\xi}_{\Lambda_n\setminus\Lambda})|. \tag{4.13}$$

Therefore,

$$\begin{aligned} &|\gamma_{\{0\}}(\xi_0|\xi) - \gamma_{\{0\}}(\xi_0|\tilde{\xi})| \\ &\quad = \lim_{n\to\infty} |\mu(\xi_0|\xi_{\Lambda_n}) - \mu(\xi_0|\tilde{\xi}_{\Lambda_n})| \leqslant \varepsilon \end{aligned} \tag{4.14}$$

yielding the continuity of $\gamma_{\{0\}}(\xi|\cdot)$.

Since μ is uniformly non-null, the one point specification $\gamma_{\{i\}}(\xi|\xi)$ actually determines a continuous specification Γ which is uniformly non-null. We finally prove that

$\mu \in \mathcal{G}(\Gamma)$. Let f be a local function, then

$$\begin{aligned}
&\int \gamma_\Lambda(f)\,\mathrm{d}\mu \\
&\quad = \lim_{n\to\infty} \int \mu(\mathrm{d}\omega) \int \mu(\mathrm{d}\sigma_\Lambda | \omega_{\Lambda_n\setminus\Lambda}) f(\sigma_\Lambda \omega_{\Lambda_n\setminus\Lambda}) \\
&\quad = \lim_{n\to\infty} \int \mathrm{d}\mu\, \mathbb{E}_\mu[f | \mathcal{F}_{\Lambda_n\setminus\Lambda}] \\
&\quad = \int f\,\mathrm{d}\mu. \qquad \square
\end{aligned} \tag{4.15}$$

For μ a probability measure on $(\Omega, \mathcal{F})$, put

$$S_\mu := \{\xi \in \Omega | \xi \text{ is 'bad' for } \mu\}. \tag{4.16}$$

From Proposition 4.2 and the proof of Proposition 4.3 we get:

Proposition 4.4. (1) $S_\mu = \emptyset \Leftrightarrow \mu$ *is Gibbsian*;
(2) $\mu(S_\mu) = 0 \Leftrightarrow \mu$ *is almost Gibbsian.*

5. Examples

In this section we discuss two examples. First of all, we consider a non-trivial convex combination of two measures which are tail trivial and mutually singular on the tail field. This will give an example of a measure which is not almost Gibbsian, nor weakly Gibbsian for a vacuum potential or 'Kozlov potential'. Second, we construct a concrete example of a measure which is weakly Gibbsian but not almost Gibbsian.

5.1. Convex combinations

Let μ_1 and μ_2 be two tail trivial probability measures which are mutually singular on the tail field, i.e. there exists a set F in the tail field such that $\mu_1(F) = 1$ and $\mu_2(F) = 0$. We also assume that there is no specification Γ for which both $\mu_1, \mu_2 \in \mathcal{G}(\Gamma)$. We can, e.g. think of $\mu = \nu_\rho$, the Bernoulli measure on $\{0,1\}^{\mathbb{Z}^d}$ with density ρ of ones, and $\mu_2 = \nu_{\rho'}$, with $0 < \rho < \rho' < 1$. In that case F is, e.g. the set of configurations with "density" ρ: $F = \{\eta \in \Omega : \liminf_{n\to\infty} 1/2n \sum_{x=-n}^{n} \eta(x) = \rho\}$.

Let $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$, $0 < \lambda < 1$ be a non-trivial convex combination of μ_1 and μ_2. It is known that for every specification Γ consistent with μ there are no points of continuity, i.e., $\Omega_\Gamma = \emptyset$. This rather drastic non-Gibbsian behavior is, however, obvious and due to the fact that every version of the conditional probabilities 'involves' a tail measurable function: for μ-a.e. η

$$\mathbb{E}_\mu[f | \mathcal{F}_{\Lambda^c}](\eta) = 1_F(\eta)\mathbb{E}_{\mu_1}[f | \mathcal{F}_{\Lambda^c}](\eta) + 1_{F^c}(\eta)\mathbb{E}_{\mu_2}[f | \mathcal{F}_{\Lambda^c}](\eta). \tag{5.1}$$

In fact, we have more: if we are given $\theta \in \Omega$, then for every specification Γ consistent with μ, we have $\mu(\Omega_\Gamma^\theta) < 1$. This follows from the observation that for a tail measurable function $\Phi:\Omega\to\mathbb{R}$ which is continuous in the direction θ in a point $\eta\in\Omega$ we have $\Phi(\eta)=\Phi(\theta)$.

So we conclude from Proposition 4.2 that the measure μ cannot be almost Gibbsian neither almost Gibbsian in a direction. Therefore, μ cannot be weakly Gibbsian with a vacuum potential nor with the 'Kozlov-potential' of Theorem 3.1 (which is essentially a resummation of the vacuum potential). From Remark 5 following Definition 4.3, we conclude that μ cannot be weakly Gibbsian for a potential U converging on a set Ω_U which is 'topologically large' (e.g. Ω_U cannot be Ω). However, we do not know whether μ is not weakly Gibbsian at all. This problem is related to the question whether or not we can represent a tail measurable function as a sum of potentials on a set of measure one, e.g. in the case of the convex combination of two Bernoulli measures on $\{0,1\}^{\mathbb{Z}}$, the question reduces to the following: does there exist a potential $U(A,\eta)$ such that $\forall i\in\mathbb{Z}$

$$\sum_{A\ni i} U(A,\eta)=\liminf_{n\to\infty} 1/2n\sum_{x=-n}^{n}\eta(x) \tag{5.2}$$

on a set $K\subset\Omega$ of η's with $\nu_\rho(K)=\nu_{\rho'}(K)=1$? Of course such a representation of a 'global' quantity in terms of 'local' quantities seems very unnatural and it is certainly not possible in a smooth way. It is, however, important for the concept of 'weakly Gibbsian' measures that these convex combinations are not 'weakly Gibbsian', because otherwise any reasonable notion of 'physical equivalence' would be ruled out in the 'weakly Gibbsian' formalism (i.e. we could have that two Gibbs measures with physically unequivalent potentials are both 'weakly Gibbsian' with the same potential).

Remark. There is also an example of a strongly mixing measure μ which gives positive weights to all non-empty cylinder events for which $S_\mu=\Omega$ and which is not weakly Gibbsian, see van den Berg, Lörinczi et al. (1997a, b).

5.2. *A weakly Gibbsian measure which is not almost Gibbsian*

The counterexample discussed here is of the form $\mu(\mathrm{d}\eta)=f(\eta)\nu_{1/2}(\mathrm{d}\eta)$, where $\nu_{1/2}$ is the Bernoulli measure on $\{0,1\}^{\mathbb{N}}$, and $f(\eta)=\exp(-H(\eta))$ is a positive density function which is discontinuous enough to ensure that μ is not almost Gibbsian. However, $H(\eta)$ will be given as a sum of potentials which converges $\nu_{1/2}$ a.s., thus making μ into a weakly Gibbs measure.

We consider $\Omega=\{0,1\}^{\mathbb{N}}$ and we define for $0<\rho<1$ (fixed) a potential

$$U([0,2n],\xi)=\xi(0)\xi(2n)\rho^{n-N_{2n}(\xi)}1_{\{N_{2n}(\xi)\leqslant n\}}, \tag{5.3}$$

where

$$N_{2n}(\xi)=\max\{j\geqslant 0|\xi(2n)\xi(2n-1)\dots\xi(2n-j)=1\} \tag{5.4}$$

and $U(A,\xi)=0$ when $A\notin\{[0,2n]:\ n\in\mathbb{N}\}$. To clarify these formulas, the interaction of the interval $[0,2n]$ is obtained as follows: count from the right endpoint the number of successive ones, say N. The interaction reaches its maximum for $N=n$ (it is then equal to 1), it becomes zero for $N>n$ and decreases in $0\leqslant N\leqslant n$ to the minimal value ρ^n in $N=0$. We define

$$H(\eta):=\sum_{n\geqslant 0} U([0,2n],\eta) \tag{5.5}$$

on the set of η's for which this sum converges. We will show in Lemma 5.1 below that $H(\eta)$ exists a.s. for the Bernoulli measure $\nu_{1/2}$ on $\{0,1\}^{\mathbb{N}}$. Henceforth, we abbreviate $\nu=\nu_{1/2}$. We will then construct μ as the measure having density $\mathrm{e}^{-H(\eta)}$ with respect to ν, and we show that for this μ the set of 'bad' configurations has positive measure.

Lemma 5.1.

$$\sum_{n\geqslant 0} U([0,2n],\eta)<\infty \qquad \nu\text{-}a.s.. \tag{5.6}$$

Proof. For $\varepsilon>0$ we have

$$\left\{\eta\colon\ \sum_n U([0,2n],\eta)=\infty\right\}\subset A_\varepsilon,$$

where

$$\begin{aligned} A_\varepsilon &:= \{\xi\in\Omega\colon\ \forall N\in\mathbb{N}\ \exists p\geqslant N\colon\ \xi(2p)\xi(2p-1)\ldots\xi(p+(\log p)^{1+\varepsilon})=1\} \\ &= \bigcap_{N\in\mathbb{N}}\bigcup_{p\geqslant N} A_{p,\varepsilon}, \end{aligned} \tag{5.7}$$

where $A_{p,\varepsilon}=\{\xi\colon\ \xi(2p)\xi(2p-1)\ldots\xi(p+(\log p)^{1+\varepsilon})=1\}$. Now we have

$$\nu(A_{p,\varepsilon})=(\tfrac{1}{2})^{p-(\log p)^{1+\varepsilon}} \tag{5.8}$$

and thus

$$\sum_{p=0}^{\infty}\nu(A_{p,\varepsilon})<\infty. \tag{5.9}$$

Therefore $\nu(A_\varepsilon)=0$. $\square$

Define the measure μ on $\{0,1\}^{\mathbb{N}}$ by

$$\int \mu(\mathrm{d}\eta)f(\eta)=\frac{\int \mathrm{e}^{-H(\eta)}f(\eta)\nu(\mathrm{d}\eta)}{\int \mathrm{e}^{-H(\eta)}\nu(\mathrm{d}\eta)}. \tag{5.10}$$

For the conditional probability $\mu(\sigma_0=1|\xi_1\ldots\xi_{2n})$ we obtain

$$\begin{aligned} \mu(\sigma_0=1|\xi_1\ldots\xi_{2n}) &= \frac{\int \mathrm{e}^{-H(1\xi_1\ldots\xi_{2n}\eta_{[0,2n]^c})}\nu(\mathrm{d}\eta)}{1+\int \mathrm{e}^{-H(1\xi_1\ldots\xi_{2n}\eta_{[0,2n]^c})}\nu(\mathrm{d}\eta)} \\ &= \frac{1}{1+[\int \mathrm{e}^{-H(1\xi\eta)}\nu(\mathrm{d}\eta)]^{-1}}. \end{aligned} \tag{5.11}$$

We first show that for n large enough and for all $\xi_1 \dots \xi_{2n}$ there exist $\xi^1_{2n+1} \dots \xi^1_{4n+1}$ and $\xi^2_{2n+1} \dots \xi^2_{4n+1}$ such that $|H(1\xi\xi^1\eta) - H(1\xi\xi^2\eta)| > \rho/2$. Indeed, choose $\xi^1_{2n+1} = 0$, $\xi^1_{2n+2} = \cdots = \xi^1_{4n} = 1$, $\xi^1_{4n+1} = 0$ and $\xi^2_{2n+1} = \cdots = \xi^2_{4n+1} = 0$. Then

$$|H(1\xi\xi^1\eta) - H(1\xi\xi^2\eta)| = \sum_{k=1}^{n} \rho^k > \frac{\rho}{2}. \tag{5.12}$$

Therefore, combining this with Eq. (5.11)

$$\begin{aligned} &|\mu(\sigma(0)=1|\xi_1 \dots \xi_{2n}\xi^1_{2n+1} \dots \xi^1_{4n+1}) - \mu(\sigma(0)=1|\xi_1 \dots \xi_{2n}\xi^2_{2n+1} \dots \xi^2_{4n+1})| \\ &\quad \geqslant \frac{1}{4}\int \nu(\mathrm{d}\eta)\exp[-H(1\xi\xi^1\eta](\mathrm{e}^{\rho/2} - 1). \end{aligned} \tag{5.13}$$

If we now restrict our ξ's to the set $\{\xi\colon H(\xi) \leqslant a\}$ which has strict positive measure for a large enough, then, for these ξ's, using Jensen's inequality

$$\int \nu(\mathrm{d}\eta)\exp(-H(1\xi\xi^1\eta)) \geqslant \exp(-(a+b+1/\rho)), \tag{5.14}$$

where $b = \int \nu(\mathrm{d}\eta)H(1\eta) < \infty$. Thus, there is a set of 'bad' configurations of strict positive μ-measure. Now it is clear that μ is weakly Gibbsian with potential U, since μ-a.s.

$$\begin{aligned} \mu(\sigma_\Lambda|\omega_{\Lambda^c}) &= \frac{\exp(-\sum_{A\subset\mathbb{Z}} U(A,\sigma_\Lambda\omega_{\Lambda^c}))}{\sum_{\sigma'_\Lambda}\exp(-\sum_{A\subset\mathbb{Z}} U(A,\sigma'_\Lambda\omega_{\Lambda^c}))} \\ &= \frac{\exp(-\sum_{A\cap\Lambda\neq\emptyset} U(A,\sigma_\Lambda\omega_{\Lambda^c}))}{\sum_{\sigma'_\Lambda}\exp(-\sum_{A\cap\Lambda\neq\emptyset} U(A,\sigma'_\Lambda\omega_{\Lambda^c}))}. \end{aligned} \tag{5.15}$$

Since there exists a set of 'bad' configurations of strict positive measure, μ is not almost Gibbsian but since by Lemma 5.1 the potential converges (absolutely) on a set of μ-measure 1, μ is weakly Gibbsian.

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